LOCALIZATION GENUS

JESPER M. MØLLER AND JÉRÔME SCHERER

Abstract. Which spaces look like an n-sphere through the eyes of the n-th Postnikov section functor and the n-connected cover functor? The answer is what we call the Postnikov genus of the n-sphere. We define in fact the notion of localization genus for any homotopical localization functor in the sense of Bousfield and Dror Farjoun. This includes exotic genus notions related for example to Neisendorfer localization, or the classical Mislin genus, which corresponds to rationalization.

Introduction

Classically the genus of a nilpotent space X of finite type, as introduced by Mislin in [16], consists of all homotopy types of nilpotent spaces Y of finite type such that the localizations $Y_{(p)}$ and $X_{(p)}$ coincide at any prime p. That is, spaces in the same genus as X cannot be distinguished from X if one looks at them through the eyes of p-localization. Another equivalent definition can be given in terms of rationalization and p-completion, [21].

We introduce in this article the notion of *localization genus*. A localization functor L in the category of spaces (or simplicial sets), as introduced by Bousfield, [3], Farjoun, [7], is a homotopy functor equipped with a natural transformation η from the identity which is idempotent up to homotopy. The main point to study such functors is that it subsumes the notions of localization at a prime or a set of primes (e. g. rationalization) and p-completion, but also Postnikov sections, Quillen's plus-construction, and other nullification or periodization functors such as $P_{B\mathbb{Z}/p}$, which plays a central role in the Sullivan conjecture, [15]. We write $\bar{L}X$ for the homotopy fiber of the natural map $\eta_X : X \to LX$. We define thus two genus sets associated to L for any simply connected CW-space X of finite type.

(1) The *extended L genus set* for X is the set $\bar{G}_L(X) = \{Y \mid LY \simeq LX,\ \bar{L}Y \simeq \bar{L}X\}$ of homotopy types Y of CW-spaces such that $LY = LX$ and $\bar{L}Y = \bar{L}X$.

(2) The *L-genus set* for X is the subset $G_L(X)$ of $\bar{G}_L(X)$ represented by CW-spaces of finite type.

Date: June 19, 2015.

2000 *Mathematics Subject Classification.* Primary 55S45; Secondary 55R15, 55R70, 55P20, 22F50.

The first author is supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) and by Villum Fonden through the project Experimental Mathematics in Number Theory, Operator Algebras, and Topology. The second author is supported by FEDER/MEC grant MTM2013-42293-P.

Our definition is motivated by the classical definition of the (completion) genus set, [16], [21, Definition 3.2], and the extended (completion) genus set studied by McGibbon in [13]. We show in fact in Proposition 1.3 that when L is rationalization, one gets back these classical notions. To illustrate our point of view we go through the computation of the extended rationalization genus $\bar{G}(S^n)$ of an odd sphere, Theorem 2.2, and we characterize in Corollary 2.4 those elements in $\bar{G}(S^n)$ corresponding to elements in the extended genus of the *abelian group* of integers, as studied by Hilton in [10].

To tackle technically harder problems we rely on Dywer and Kan's classifiyng space for towers of fibrations, [6], a tool which has proven to be handy in similar situations, [17]. This allows us in particular to do explicit computations of Postnikov genus sets for spheres and complex projective spaces.

Theorem 6.6. *The extended Postnikov genus set $\overline{G}_{[n]}(S^n)$ of homotopy types of spaces Y such that $Y[n] \simeq K(\mathbb{Z}, n)$ and $Y\langle n\rangle \simeq S^n\langle n\rangle$ is uncountable, in bijection with $\prod_p \mathbb{N}_+$, where the product is taken over all primes.*

We also present in Section 4 a computation related to Neisendorfer's functor, [18], and the Sullivan conjecture. The localization genus computations show combined features of the space one focuses on and the chosen localization functor. The notion of genus quantifies in which sense it is (not) sufficient to consider a given space locally, through the eyes of a localization functor L and the associated fiber $\bar{L}$.

Acknowledgements. This project started during a visit of the first author at the Universitat Autònoma de Barcelona in 2008 and was rebooted during a visit of the second author at the Centre for Symmetry and Deformation in Copenhagen seven years later.

1. Genus and extended genus

Let L be a homotopical localization functor, i.e. a coaugmented and idempotent homotopy functor in the category of spaces. It is sometimes more convenient to work in the Quillen equivalent category of simplicial sets, in particular when one needs models for mapping spaces. We will clearly say so when we do so. In practice localization functors arise as follows. To any map f one associates a functor L_f which inverts f in a universal way, [7] and [3]. Clever choices for the map yield homological localization, localization at a set of primes such as rationalization, Quillen's plus construction, Postnikov sections, etc.

The homotopy fiber of the coaugmentation $X \to LX$ is $\overline{L}X$.

Definition 1.1. Let L be a localization functor and X a simply connected CW-complex of finite type.

- The *extended-L genus set* for X is the set

$$\overline{G}_L(X) = \{Y \mid LY \simeq LX,\ \overline{L}Y \simeq \overline{L}X\}$$

of homotopy types Y of CW-spaces such that $LY = LX$ and $\overline{L}Y = \overline{L}X$.
- The *L-genus set* for X is the subset $G_L(X)$ of $\overline{G}_L(X)$ represented by CW-complexes of finite type.

The reason for this "generic" terminology comes from the relationship with the classical notion of genus. Recall that Mislin's definition, [16], is given in terms of localization at primes: Two spaces X and Y belong to the same genus set if their localizations $X_{(p)}$ and $Y_{(p)}$ are homotopy equivalent at every prime p. This is a stronger requirement than merely asking for equivalent p-completions $X^\wedge_p$ and $Y^\wedge_p$, for any prime p, and equivalent rationalizations X_0 and Y_0, as shown for example by Belfi and Wilkerson in [2]. We will focus on the completion genus set as in [21, Definition 3.5] and the extended completion genus set, [13]. We denote by $X^\wedge$ the product of all p-completions.

Definition 1.2. Let X be a simply connected CW-complex of finite type.

- The *extended genus set* of X is the set $\overline{G}(X)$ of homotopy types of CW-complexes Y such that $Y^\wedge = X^\wedge$ and $Y_0 = X_0$.
- The *genus set* of X is the subset $G(X)$ of $\overline{G}(X)$ represented by CW-complexes Y of finite type.

We show now that the classical completion genus coincides with our localization genus, when the chosen localization functor L is rationalization. Since we restrict our attention to simply connected spaces here, one can choose the map f to be the wedge of degree p maps on the 2-sphere, taken over all primes p. Then the Bousfield localization functor L_f is rationalization on simply connected spaces. When $LX = X_0$ is rationalization, we write X_τ for $\overline{L}X$, the fibre of the rationalization map $X \to X_0$ and call it the *torsion space* of X.

Proposition 1.3. *Let X be a simply connected CW-complex of finite type. The extended rationalization-genus set*

$$\overline{G}_0(X) = \{Y \mid Y_0 \simeq X_0,\ Y_\tau \simeq X_\tau\} = \{Y \mid Y_0 \simeq X_0,\ Y^\wedge \simeq X^\wedge\}$$

is the extended genus set $\overline{G}(X)$ and the rationalization-genus set $G_0(X)$ coincides with the classical genus set $G(X)$.

Proof. To see this, note that $X_\tau \simeq Y_\tau$ if and only if $X^\wedge \simeq Y^\wedge$. Indeed, if we complete the fibration $X_\tau \to X \to X_0$ (by the nilpotent fibration Lemma [4, II.4.8]) we see that $(X_\tau)^\wedge = X^\wedge$, and Sullivan's arithmetic square, [4, VI.8.1], shows that X_τ is (also) the fibre of $X^\wedge \to (X^\wedge)_0$. ∎

Let us finally remark that all spaces in $G(X)$ are finite complexes when X is a finite complex. This comes from the fact that, when X is of finite type, the integral homology groups of any space in the genus set of X are those of X. There is thus a Moore-Postnikov decomposition of such a space as successive homotopy cofibers of maps between (finite) Moore spaces, see for example [9, Chapter 8].

2. The extended rationalization genus of an odd sphere

In this section we turn our attention to a concrete example and propose an explicit computation of the rationalization-genus for odd spheres. Let n be an odd natural number. The extended rationalization genus set of the odd-dimensional sphere S^n is according to [13, Theorem 3] an uncountable set. We offer an explicit description of this extended genus set and identify the elements known as pseudo-spheres. For this we will need some elementary abelian group theory.

Any torsion free abelian group A of rank one can be seen, up to isomorphism, as a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. For each prime p, let $k_p(A) = \max\{r \geq 0 \mid 1 \in p^r A\}$ denote the *height* of 1 at p. The *height sequence* of A is the sequence $\Pi(A) = (k_p(A))_p$ of non-negative (or infinite) integers. Two sequences (k_p) and (m_p) are similar if the sum of the differences $|k_p - m_p|$ is finite. This means that the sequences differ in only a finite number of primes, and have ∞ in the same coordinate. A *type* is a similarity class of sequences. As explained in [8], or [19, Theorem 10.47], isomorphism types of torsion free abelian groups of rank one are in bijection with types.

We start now with the fibration $S^n_\tau \to S^n \to (S^n)_0$ and will use that, since n is odd, $(S^n)_0 \simeq K(\mathbb{Q}, n) \simeq M(\mathbb{Q}, n)$ and $S^n_\tau \simeq M(\mathbb{Q}/\mathbb{Z}, n-1)$. For any space Y in the extended rationalization genus of S^n we have a fibration sequence $M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y \to K(\mathbb{Q}, n)$. The homotopy long exact sequence yields an exact sequence

$$0 \to \pi_n Y \to \mathbb{Q} \xrightarrow{\partial} \mathbb{Q}/\mathbb{Z} \to \pi_{n-1} Y \to 0\,.$$

The idea is that the connecting homomorphism ∂ determines the homotopy type of Y.

Lemma 2.1. *There is a bijection between isomorphism types of torsion free abelian groups of rank one and the double coset $\mathbb{Q}^\times \backslash \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})/\hat{\mathbb{Z}}$, where the units in $\mathbb{Q}$ act by pre-composition and the automorphisms $\hat{\mathbb{Z}}$ of $\mathbb{Q}/\mathbb{Z}$ by post-composition.*

Proof. We define two maps. The first $\alpha : \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \{A \mid A \text{ torsion free abelian of rank one}\}$ sends a homomorphism ∂ to its kernel. The second one we call β. Given a subgroup A of $\mathbb{Q}$, let J be the subset of all primes consisting of those primes p for which $n_p \neq \infty$. Then the quotient $\mathbb{Q}/A$ is isomorphic to $\oplus_{p\in J} \mathbb{Z}_{p^\infty}$ and β sends A to the composite

$$\mathbb{Q} \to \mathbb{Q}/A \xrightarrow{\cong} \oplus_{p\in J} \mathbb{Z}_{p^\infty} \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

Clearly $\alpha \circ \beta$ sends a torsion free abelian group of rank one A to a subgroup of $\mathbb{Q}$ which is isomorphic to A. Moreover, given a homomorphism $\partial : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, the image $\beta(\operatorname{Ker} \partial)$ coincides with ∂ up to

an isomorphism of $\mathbb{Q}$ (corresponding to the choice of an inclusion $\alpha(\partial) \subset \mathbb{Q}$) and an isomorphism of $\mathbb{Q}/\mathbb{Z}$ (corresponding to the choice of an isomorphism $\mathbb{Q}/\operatorname{Ker}\partial \cong \oplus_{p\in J}\mathbb{Z}_{p^\infty}$). This proves the lemma. ∎

We proceed now with the construction for any homomorphism $\partial : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ of a space $Y(\partial)$ in the extended genus $\overline{G}(S^n)$ realizing this homomorphism as connecting map in the homotopy long exact sequence of the fibration $M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y(\partial) \to K(\mathbb{Q}, n)$. There is a bijection $[M(\mathbb{Q}, n-1), M(\mathbb{Q}/\mathbb{Z}, n-1)] \cong \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ and so there exists up to homotopy a unique map $\Delta : M(\mathbb{Q}, n-1) \to M(\mathbb{Q}/\mathbb{Z}, n-1)$ such that $\pi_{n-1}(\Delta) = \partial$. We define $Y(\partial)$ to be the homotopy cofiber of Δ. We have therefore a cofibration sequence

$$M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y(\partial) \to M(\mathbb{Q}, n)$$

which is seen to be a fibration sequence as well, for example by an elementary Serre spectral sequence argument (a complete characterization of such sequences has been obtained by Alonso, [1], see also Wojtkowiak's [22]).

Theorem 2.2. *The extended genus set $\overline{G}(S^n)$ is in bijection with the set of isomorphism classes of torsion free abelian groups of rank one.*

Proof. Given a torsion free abelian group A of rank one we get a homomorphism $\partial : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ from Lemma 2.1 and construct as above the space $Y(\partial)$. It realizes ∂ as connecting homomorphism in the homotopy long exact sequence, and its kernel is a group isomorphic to A. To show that we have indeed a bijection we must prove that the connecting homomorphism determines the homotopy type of $Y \in \overline{G}(S^n)$. Let ∂ be the connecting homomorphism for such a space Y and let us compare Y and $Y(\partial)$. We have a map $i : M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y$ by definition of the extended genus set and consider the composite $i \circ \Delta : M(\mathbb{Q}, n-1) \to Y$, where Δ is, as above, the the unique map, up to homotopy, realizing ∂. It coincides thus, up to homotopy, with the composite of the Postnikov section $M(\mathbb{Q}, n-1) \to K(\mathbb{Q}, n-1)$ and the homotopy fiber inclusion of $K(\mathbb{Q}, n-1)$ in the total space of the map $M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y$. This shows that the composite $i \circ \Delta$ is homotopically trivial. Therefore the map i factors through the homotopy cofiber of Δ, i.e. $Y(\partial)$.

We have thus constructed a map $Y(\partial) \to Y$ which induces equivalences on rationalizations and torsion spaces. It is hence an equivalence as well. ∎

In the final part of the section we restrict our attention to the $(n-1)$-connected members of the extended genus set of an odd sphere. We begin with a review of Hilton's investigations of the extended genus set of $\mathbb{Z}$ and groups of pseudo-integers [10, 11].

Definition 2.3. [10] A subgroup of the full rational group $\mathbb{Q}$ is a *group of pseudo-integers* if it contains $\mathbb{Z}$ but not $\mathbb{Z}[1/p]$ for any prime p.

Since a group of pseudo-integers is a torsion free abelian group of rank one, in the terminology introduced at the beginning of the section, it is characterized by its type, which consists in only finite integers k_p. This subset of torsion free abelian group of rank one has been studied by Hilton in [10, Theorem 2.3, 2.4].

According to [10, Corollary 2.5], the extended genus set $\overline{G}(\mathbb{Z})$ of $\mathbb{Z}$, consisting of isomorphism classes of (not necessarily finitely generated) abelian groups H such that $H \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}$ for all primes p, is the set of isomorphism classes of pseudo-integers. In other words, $\bar{G}(\mathbb{Z})$ is the set of isomorphism classes of torsion free abelian groups of rank 1 of ∞-free types [8, §42].

Corollary 2.4. *The set of $(n-1)$-connected spaces in $\overline{G}(S^n)$ is in bijection with $\overline{G}(\mathbb{Z})$. The correspondence is given by $Y \mapsto \pi_n(Y)$.*

Proof. The spaces in the extended genus of S^n which are $(n-1)$-connected are characterized by the fact that the connecting homomorphism ∂ is surjective. In other words its kernel is a group of pseudo-integers. We conclude by Theorem 2.2. ∎

3. A formula to compute localization genera

In the previous section we have been able to establish a complete and explicit list of all homotopy types in the extended rationalization genus of an odd sphere. In general, for arbitrary spaces and arbitrary localization functor, this is not to be expected. Following the approach of Dwyer, Kan, and Smith in [6] to classify towers of fibrations, we propose in this section a formula which we use later on to perform computations of "Postnikov" and "Neisendorfer" genus. We start with the necessary background from [6].

Let G be a space and consider the functor Φ which sends an object of *Spaces* $\downarrow B\,\mathrm{aut}(G)$, i.e. a map $t : X \to B\,\mathrm{aut}(G)$, to the twisted product $X \times_t G$. Dwyer, Kan, and Smith describe a right adjoint Ψ in [6, Section 4]. They find first a model for $\mathrm{aut}(G)$ which is a (simplicial) group and thus acts on the left on $\mathrm{map}(G, Z)$ for any space Z. This induces a map $r : B\,\mathrm{aut}(G) \to B\,\mathrm{aut}(\mathrm{map}(G, Z))$. The functor Ψ sends then Z to the projection map from the twisted product $B\,\mathrm{aut}(G) \times_r \mathrm{map}(G, Z) \to B\,\mathrm{aut}(G)$. This allows right away to construct a classifying space for towers, in our case they will be of length 2.

Theorem 3.1. Dwyer, Kan, Smith, [6]. *The classifying space for towers of the form $Z \xrightarrow{q} Y \xrightarrow{p} X$, where the homotopy fiber of p is G and that of q is H, is $B\,\mathrm{aut}(G) \times_r \mathrm{map}(G, B\,\mathrm{aut}(H))$.*

Remark 3.2. Working with the Dwyer-Kan-Smith model means that we deal with simplicial sets. The comparison with spaces is via the singular complex and geometric realization. Since the realization of a simplicial set is a CW-complex, all spaces we construct here have obviously the homotopy type of a CW-complex.

There is an interesting consequence of Theorem 3.1, which can be compared with our former result about the classifying space of the monoid of self-equivalences of a two-stage Postnikov piece, [17, Theorem 5.3].

Theorem 3.3. *Let L be a localization functor and X be a space such that $L\overline{L}X$ is contractible. There is then a bijection between $G_L(X)$ and the set*

$$[LX, B\,\mathrm{aut}(\bar{L}X)]/\,\mathrm{Aut}(LX)$$

of orbits for the action of the group $\mathrm{Aut}(LX)$ *on the set* $[LX, B\,\mathrm{aut}(\bar{L}X)]$.

Proof. Equivalence classes of towers of fibrations $Z \to Y \to *$ with successive fibres LX and $\bar{L}X$ are classified by the set of path components of $B = B\,\mathrm{aut}(LX) \times_r \mathrm{map}(LX, B\,\mathrm{aut}(\bar{L}X))$. It is clear that any such space Z fits in a fibration $\bar{L}X \to Z \to LX$, but we must show that the map $Z \to LX$ coincides with the localization map for Z. Fibrewise localization, [7, Theorem 1.F.1], yields a natural transformation between fibration sequences

$$\begin{tikzcd}
\overline{L}X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & LX \arrow[d, equal] \\
L\overline{L}X \arrow[r] & \overline{Z} \arrow[r] & LX
\end{tikzcd}$$

By assumption $L\overline{L}X$ is contractible and since by construction the map $Z \to \overline{Z}$ is an L-equivalence it must coincide with the localization map $Z \to LX$. ∎

The assumption that $L\overline{L}X$ is contractible is restrictive if we would require this for all spaces X. This would amount to imposing that the localization functor L is a so-called *nullification* functor, i.e. a homotopical localization functor associated to a map of the form $A \to *$ such as a Postnikov section – when A is a sphere, [7, 1.A.6] – or Quillen's plus construction. We impose this condition however on a single space, and this happens sometimes for localization functors that are not nullifications. When X is a simply connected space of finite type and L is rationalization, p-completion, or completion, then $L\overline{L}X \simeq *$. These are the main examples of interest in this note.

According to Theorem 3.3, one can often compute a localization genus set via a Dwyer-Kan-Smith type formula.

Corollary 3.4. *The classical extended genus set of a simply connected space X of finite type is given by $\overline{G}(X) = [X_0, B\,\mathrm{aut}(X_\tau)]/\,\mathrm{Aut}(X_0)$.*

4. Neisendorfer genus and Postnikov genus

Let P be the nullification functor [7, 1.A.4] with respect to the wedge $\bigvee B\mathbb{Z}/p$ taken over all primes p. The unexpected effect of P on highly connected covers of finite spaces has been first studied by Neisendorfer, [18]. It relies on Miller's solution to the Sullivan conjecture, [15]. The

name *Neisendorfer's functor* is commonly used for $P_{B\mathbb{Z}/p}$ followed by completion at the prime p. This composition of two localization functors however is not itself a localization functor because it fails to be idempotent in general. Therefore we use here the name Neisendorfer's functor for localization with respect to the maps $c\colon \bigvee B\mathbb{Z}/p \to *$ and a wedge of universal mod p homology equivalences h. Thus L_h coincides with profinite completion on nilpotent spaces and $\hat{P}X = L_{c\vee h}X$ often agrees with $(PX)^\wedge$, e.g. when the latter space is already $B\mathbb{Z}/p$-null.

Theorem 4.1. [18, Theorem 4.1] *Let X be a simply connected finite complex with $\pi_2(X)$ finite and let $n \geq 1$ be a natural number. Then*

$$P(X\langle n\rangle) = \operatorname{holim}(X \to X_0 \leftarrow X\langle n\rangle_0)$$

so that $P(X\langle n\rangle)_\tau \simeq X_\tau$ and $P(X\langle n\rangle)^\wedge \simeq X^\wedge$.

As a direct consequence we compute the Neisendorfer genus of a finite complex. In a given localization genus set we fix the localization and the fiber of the localization map. Thus, even if all highly connected covers of a finite complex have the same Neisendorfer localization as the original complex, they do not belong to the (extended) genus set since the fibers of the localization maps fail to agree.

Corollary 4.2. *Let X be a simply connected finite complex with $\pi_2 X$ finite. Then $\overline{G}_P(X) = G_P(X) = \{X\}$.*

Proof. Applying Theorem 4.1 to X with $n = 1$ we see that $PX \simeq X$. ∎

For us the following consequences will be important. In particular point (3) will help compute the monoids of self-equivalences which appear in the formula from Corollary 3.4.

Corollary 4.3. *Let X be a simply connected finite complex with $\pi_2(X)$ finite. Suppose that $\pi_{>N}(X) \otimes \mathbb{Q} = 0$ for some integer N. Then*

(1) $P(X\langle N\rangle) \simeq X_\tau$
(2) [14] *There are weak equivalences*

$$\operatorname{aut}_*(X\langle N\rangle) \simeq \operatorname{aut}_*(X_\tau) \simeq \operatorname{aut}_*(X^\wedge) \simeq \operatorname{aut}_*(X^\wedge\langle N\rangle)$$

of topological monoids of pointed self homotopy equivalences

(3) *The obvious map $G(X) \to G(X[N])$ is bijective*

Proof. Since $X\langle N\rangle_0$ is contractible we deduce from Theorem 4.1 that $P(X\langle N\rangle) = X_\tau$, which proves (1). Hence starting with $X\langle N\rangle$ we see that we get first X_τ by applying P, then $X^\wedge$ by applying completion, and finally $X\langle N\rangle$ by taking the N-connected cover. As this last functor is a pointed one (it can be seen for example as cellularization with respect to the sphere S^{N+1}, [7,

Example 2.D.6]), we obtain a chain of weak homotopy equivalences $\mathrm{aut}_*(X\langle N\rangle) \simeq \mathrm{aut}_*(X_\tau) \simeq \mathrm{aut}_*(X^\wedge) \simeq \mathrm{aut}_*(X^\wedge\langle N\rangle)$. This shows (2).

Wilkerson's double coset formula for the genus set [21, Theorem 3.8] exhibits $G(X)$ as double coset of the so-called π_*-continuous self-equivalences of $(X^\wedge)_0$ under the left action of $\mathrm{aut}(X^\wedge)$ and the right action of $\mathrm{aut}(X_0)$. Clearly X and the Postnikov section $X[N]$ are rationally identical by assumption. Also the groups of self-equivalences are the same for X and its completion by part (2) of this corollary (which is for the pointed version but the spaces here are simply connected). Thus all ingredients in Wilkerson's formula are identical for X and $X[N]$, which shows that the map $G(X) \to G(X[N])$ is bijective. ∎

We note that $G(X[N])$ can be computed from Zabrodsky's exact sequence when X_0 is an H-space [23, 25], [13, Theorem 4].

We turn now to our first computation of Postnikov genus. If $f : S^{N+1} \to *$ is the constant map, then L_f is a functorial N-th Postnikov section, $L_f X \simeq X[N]$, and $\overline{L}_f$ is a functorial N-connected cover, $\bar{L}_f X \simeq X\langle N\rangle$.

Definition 4.4. The *N-th Postnikov section genus* of a space X is the set $G_L(X)$ when L is localization with respect to $f : S^{N+1} \to *$. We write $G_{[N]}(X)$ for this set.

Hence a space Y belongs to the extended genus $\overline{G}_{[N]}(X)$ if its N-th Postnikov section $Y[N]$ coincides with $X[N]$ and its N-connected cover $Y\langle N\rangle$ coincides with $X\langle N\rangle$.

Theorem 4.5. *Let X be a simply connected finite complex with $\pi_2(X)$ finite and suppose that $\pi_{>N}(X) \otimes \mathbb{Q} = 0$ for some integer N. Then the only finite CW-complex in $G_{[N]}(X)$ is X*

Proof. If Y belongs to $G_{[N]}(X)$, then $Y\langle N\rangle \simeq X\langle N\rangle$ and $Y[N] \simeq X[N]$. Thus, if Y is finite, $Y^\wedge \simeq P(Y\langle N\rangle)^\wedge \simeq P(X\langle N\rangle)^\wedge \simeq X^\wedge$ by Theorem 4.1 and $Y_0 \simeq Y[N]_0 \simeq X[N]_0 \simeq X_0$ showing that $Y \in G(X)$. But X and Y have the same image under the injective map $G(X) \to G(X[N])$ (by the previous theorem) so $X \simeq Y$. ∎

Remark 4.6. Point (4) above tells us that there are very few finite CW-complexes in a Postnikov genus set. However there are many infinite CW-complexes of finite type in the extended genus set. For example when $X = S^3$ and L is the third Postnikov section, the space $K(\mathbb{Z}, 3) \times S^3\langle 3\rangle$ is obviously in the L-genus of S^3. We will come back to this kind of example with a detailed computation in the next section.

Remark 4.7. It is not always true that there is a single finite complex in the Postnikov genus of a finite complex X. Let us consider for example the space $S^2 \times S^5$ and the functor L is chosen to be the second Postnikov section. Then $(S^2 \times S^5)[2] \simeq K(\mathbb{Z}, 2)$ and $(S^2 \times S^5)\langle 2\rangle \simeq S^3 \times S^5$. It is easy to see that the space $\mathbb{C}P^2 \times S^3$ also belongs to $G_{[2]}(S^2 \times S^5)$. Of course the condition of the corollary are not fulfilled since neither $\pi_2 S^2$, nor $\pi_3 S^2$, are torsion.

It would be interesting to construct similar examples with higher Postnikov sections and at least 2-connected spaces, so the π_2 assumption is trivially fulfilled.

5. Self-equivalences of connected covers of a sphere

Our next goal will be to determine the n-th Postnikov genus of an odd sphere S^n with $n \geq 3$. This will be done by using Theorem 3.3, which involves the computation of the space of self-equivalences of the n-th connected cover $S^n\langle n\rangle$. This section prepares the terrain for the genus computation in the next section and focuses on handy properties of $\operatorname{aut}(S^n\langle n\rangle^\wedge_p)$.

We write $X^\wedge_p$ for the p-completion of X. Since $S^n\langle n\rangle$ is a torsion space, it is weakly equivalent to the product of its p-completions $S^n\langle n\rangle^\wedge_p$. Now

$$\operatorname{map}(S^n\langle n\rangle, S^n\langle n\rangle) \simeq \prod_p \operatorname{map}(S^n\langle n\rangle, S^n\langle n\rangle^\wedge_p) \simeq \prod_p \operatorname{map}(S^n\langle n\rangle^\wedge_p \times \prod_{q\neq p} S^n\langle n\rangle^\wedge_q, S^n\langle n\rangle^\wedge_p)$$

But since $S^n\langle n\rangle^\wedge_p$ is p-complete and $(\prod_{q\neq p} S^n\langle n\rangle^\wedge_q)^\wedge_p$ is contractible, we see that this mapping space is weakly equivalent to $\prod_p \operatorname{map}(S^n\langle n\rangle^\wedge_p, S^n\langle n\rangle^\wedge_p)$. Therefore the subspace of self-equivalences also splits as a product

$$\operatorname{aut}(S^n\langle n\rangle) \simeq \prod_p \operatorname{aut}(S^n\langle n\rangle^\wedge_p).$$

Because of the formulas in Theorem 3.3 we wish to understand certain mapping spaces into $B\operatorname{aut}(S^n\langle n\rangle^\wedge_p)$ and start with a few elementary lemmas.

Lemma 5.1. *The space $B\operatorname{aut}_*((S^n)^\wedge_p) \simeq B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)$ is $\Sigma B\mathbb{Z}/p^k$-local for any $k \geq 1$.*

Proof. From Corollary 4.3.(2) we know that the spaces of pointed self-homotopy equivalences of the p-completed sphere and its n-connected cover coincide. By adjunction the pointed mapping space $\operatorname{map}_*(\Sigma B\mathbb{Z}/p, B\operatorname{aut}_*((S^n)^\wedge_p))$ is equivalent to $\operatorname{map}_*(B\mathbb{Z}/p, \operatorname{aut}_*((S^n)^\wedge_p))$. But $\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)$ consists of certain components of $\operatorname{map}_*((S^n)^\wedge_p, (S^n)^\wedge_p) \simeq \operatorname{map}_*(S^n, (S^n)^\wedge_p) = \Omega^n(S^n)^\wedge_p$, and all components of this iterated loop space have the same homotopy type, and they are $B\mathbb{Z}/p$-local by Miller's Theorem [15]. ∎

The following is much easier to prove for the connected cover than for the p-completed sphere itself.

Lemma 5.2. *The space $B\operatorname{aut}_*((S^n)^\wedge_p) \simeq B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)$ is $K(\mathbb{Z}[1/p], 2)$-local.*

Proof. Since $K(\mathbb{Z}[1/p], 2)$ is $S^2[1/p]$-cellular it is sufficient to prove that the classifying space is $S^2[1/p]$-local. We use the usual telescopic model for this localized sphere, i.e. the homotopy colimit of the diagram $S^2 \xrightarrow{p} S^2 \xrightarrow{p} \dots$. Hence we have weak equivalences

$$\operatorname{map}_*(S^2[1/p], B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)) \simeq \operatorname{holim} \operatorname{map}_*(S^2, B\operatorname{aut}_*(X_p)) = \operatorname{holim} \Omega \operatorname{aut}_*(S^n\langle n\rangle^\wedge_p).$$

The homotopy groups of this inverse limit are all trivial since the towers we consider here consist of finite p-groups and multiplication by p (in particular all $\lim^1$ terms vanish). ∎

Even though we are not so sure whether $B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)$ is p-complete, the previous lemmas allow us to understand maps out of $K(\mathbb{Z},n)$.

Proposition 5.3. *For any $m \geq 3$ the space $B\operatorname{aut}_*((S^n)^\wedge_p) \simeq B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p)$ is $K(\mathbb{Z},m)$-local. In particular it is $K(\mathbb{Z},n)$-local.*

Proof. The space $K(\mathbb{Z}_{p^\infty}, m-2)$ is $\bigvee B\mathbb{Z}/p^k$-cellular, being a telescope of $K(\mathbb{Z}/p^k, m-2)$'s which are cellular. Hence $K(\mathbb{Z}_{p^\infty}, m-1)$ is $\bigvee \Sigma B\mathbb{Z}/p^k$-cellular by the commutation rule of cellularization with respect to loop spaces, [7, Theorem 3.A.2]. Lemma 5.1 implies then that $\operatorname{map}_*(K(\mathbb{Z}_{p^\infty}, m-1), B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p))$ is contractible. Zabrodsky's Lemma [5, Proposition 3.4] produces then an equivalence between $\operatorname{map}_*(K(\mathbb{Z},m), B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p))$ and $\operatorname{map}_*(K(\mathbb{Z}[1/p],m), B\operatorname{aut}_*(S^n\langle n\rangle^\wedge_p))$, which is contractible by Lemma 5.2. ∎

We point out that the argument with a wedge is only necessary for $m = 3$. For any larger value of m we could have gone through the same proof with $\Sigma B\mathbb{Z}/p$.

6. The extended Postnikov genus of an odd sphere

We come now to our most sophisticated computation. We wish to determine the extended Postnikov genus $\overline{G}_{[n]}(S^n)$ when n is odd and $X[n]$ is the n-th Postnikov section of X. In other words, we wish to understand how many spaces Y are extensions of $K(\mathbb{Z},n)$ by $S^n\langle n\rangle$, i.e. how many spaces look like a sphere S^n through the eyes of the n-th Postnikov section *and* the n-th connected cover functors. We write S^n_p for the fiberwise p-completion of S^n sitting in the fibration $S^n\langle n\rangle^\wedge_p \to S^n_p \to K(\mathbb{Z},n)$. By Theorem 3.3 and the previous section we know that

$$\overline{G}_{[n]}(S^n_p) = [K(\mathbb{Z},n), B\operatorname{aut}(S^n\langle n\rangle^\wedge_p)]/\{\pm 1\}, \qquad \overline{G}_{[n]}(S^n) = [K(\mathbb{Z},n), \prod_p B\operatorname{aut}(S^n\langle n\rangle^\wedge_p)]/\{\pm 1\}$$

where -1 acts on the integers by changing the sign. The following corollary should certainly be compared to Zabrodsky's [24, Corollary C'], where he deals with locally finite homotopy groups in the source.

Lemma 6.1. *For any $m \geq 3$ the space $B\operatorname{aut}((S^n)^\wedge_p)$ is $K(\mathbb{Z},m)$-local. In particular it is $K(\mathbb{Z},n)$-local.*

Proof. Consider the universal fibration $(S^n)^\wedge_p \to B\operatorname{aut}_*((S^n)^\wedge_p) \to B\operatorname{aut}((S^n)^\wedge_p)$. Since $(S^n)^\wedge_p$ is $B\mathbb{Z}/p$-local, it is also $K(\mathbb{Z}_{p^\infty}, m-1)$-local, i.e. $\operatorname{map}_*(K(\mathbb{Z}_{p^\infty}, m-1), (S^n)^\wedge_p)$ is contractible. Likewise $\operatorname{map}_*(K(\mathbb{Z}[1/p], m-1), (S^n)^\wedge_p) \simeq *$. Thus $(S^n)^\wedge_p$ is $K(\mathbb{Z},m)$-local. Next, we know that $B\operatorname{aut}_*((S^n)^\wedge_p)$ is $K(\mathbb{Z},m)$-local by Proposition 5.3. Hence, by Zabrodsky's Lemma [12, Lemma 4] we infer that $\operatorname{map}_*(K(\mathbb{Z},m), B\operatorname{aut}((S^n)^\wedge_p))$ is contractible. ∎

We compare now the classifying spaces $B\operatorname{aut}((S^n)^\wedge_p)$ and $B\operatorname{aut}(S^n\langle n\rangle^\wedge_p)$.

Proposition 6.2. *The space* $\operatorname{map}_*(K(\mathbb{Z},n), B\operatorname{aut}(S^n\langle n\rangle_p^\wedge))$ *is homotopically discrete with* $\mathbb{Z}_p^\wedge$ *components.*

Proof. Since there is a weak equivalence $B\operatorname{aut}_*(S^n\langle n\rangle_p^\wedge) \simeq B\operatorname{aut}_*((S^n)_p^\wedge)$ by Corollary 4.3.(2), we have a fibration $K(\mathbb{Z}_p^\wedge, n) \to B\operatorname{aut}(S^n\langle n\rangle_p^\wedge) \to B\operatorname{aut}((S^n)_p^\wedge)$. The proposition follows now from the previous lemma. ∎

We finally arrive at the unpointed mapping space.

Theorem 6.3. *The set of components of the space* $\operatorname{map}(K(\mathbb{Z},n), B\operatorname{aut}(S^n\langle n\rangle_p^\wedge))$ *is* $\mathbb{Z}_p^\wedge/(\mathbb{Z}_p^\wedge)^\times$. *It is in particular an infinite, countable set. Explicitely,* $\overline{G}_{[n]}(S^n_p)$ *is in bijection with the set* $\mathbb{N}_+$ *of natural numbers with a disjoint base point* $*$.

Proof. The homotopy fiber of the evaluation $\operatorname{map}(K(\mathbb{Z},n), B\operatorname{aut}(S^n\langle n\rangle_p^\wedge)) \to B\operatorname{aut}(S^n\langle n\rangle_p^\wedge)$ is homotopically discrete and identifies with $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_p^\wedge)$ by the last proposition. Moreover the fundamental group $\pi_1 B\operatorname{aut}(S^n\langle n\rangle_p^\wedge) \cong \pi_0 \operatorname{aut}(S^n\langle n\rangle_p^\wedge)$ coincides with $\pi_1 B\operatorname{aut}_*((S^n)_p^\wedge) \cong (\mathbb{Z}_p^\wedge)^\times$, the p-adic units. Their action on the p-adic integers comes from the natural action on $\pi_n(S^n)_p^\wedge$. Thus the components of the mapping space we are looking at is the quotient $\mathbb{Z}_p^\wedge/(\mathbb{Z}_p^\wedge)^\times$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $\mathbb{N}_+$ the union of $\mathbb{N}$ with a disjoint base point $*$. The quotient $\mathbb{Z}_p^\wedge/\mathbb{Z}_p^\times$ is in bijection with the set $\mathbb{N}_+$ because any non-zero p-adic integer can be uniquely written as $p^k u$ where $k \in \mathbb{N}$ and u is a unit [20]. The extended genus set has been identified as the quotient of this set under the action of ± 1. However, since -1 in $\mathbb{Z}$ is sent in the p-adics to a unit, there are no further identifications. ∎

Construction 6.4. Here is an explicit way to construct the countable set $\overline{G}_L(S^n_p)$ of spaces Y with $\pi_n Y \cong \mathbb{Z}$ and $Y\langle n\rangle \simeq S^n\langle n\rangle_p^\wedge$.

The fibration $S^n\langle n\rangle_p^\wedge \to S^n_p \to K(\mathbb{Z},n)$ is classified by a map $c\colon K(\mathbb{Z},n) \to B\operatorname{aut}(S^n\langle n\rangle_p^\wedge)$. The proof of Theorem 6.3 shows that there is a bijection

$$\mathbb{N}_+ \to [K(\mathbb{Z},n), B\operatorname{aut}(S^n\langle n\rangle_p^\wedge)]/\{\pm 1\}$$

taking $*$ to the constant map and the nonnegative integer $k \in \mathbb{N}$ to $c \circ p^k$.

Define the space $Y_{p,*}$ to be $S^n\langle n\rangle_p^\wedge \times K(\mathbb{Z},n)$ and $Y_{p,k}$, $k \in \mathbb{N}$, to be the homotopy pull-back of $K(\mathbb{Z},n) \xrightarrow{p^k} K(\mathbb{Z},n) \leftarrow S^n_p$, or, equivalently, the homotopy fibre of $S^n_p \to K(\mathbb{Z},n) \to K(\mathbb{Z}/p^k, n)$, where the second map is reduction mod p^k. The bijection is then given by

$$\begin{array}{ccc} \mathbb{N}_+ & \longrightarrow & \overline{G}_{[n]}(S^n_p) \\ k & \longmapsto & Y_{p,k} \end{array}$$

We show now that one can detect which fake partially p-completed sphere one is considering by a simple cohomological computation. We will be more precise in the proof.

Proposition 6.5. *Ordinary cohomology distinguishes all elements in the extended Postnikov genus set $\overline{G}_{[n]}(S^n_p)$.*

Proof. By Construction 6.4 these fake partially completed spheres fit into a tower of fibrations

$$\cdots \to Y_{p,k+1} \xrightarrow{f} Y_{p,k} \to \cdots \to Y_{p,1} \xrightarrow{f} Y_{p,0} = S^n_p$$

with fibers $K(\mathbb{Z}/p, n-1)$. Let ι_k denote a generator of $H^n(Y_{p,k};\mathbb{Z}) \cong \mathbb{Z}$, chosen in such a way that the image of ι_k under f^* is $p\iota_{k+1}$.

At the prime 2 the algebra structure is sufficient to distinguish the fakes. Notice that the mod 2 reduction of ι_1 is a polynomial generator detected in the mod 2 cohomology of $K(\mathbb{Z}/2, n-1)$, but ι_0 is an exterior generator. Therefore $(2^k\iota_k)^2 = 0$, but $(2^{k-1}\iota_k)^2 \neq 0$. This shows that if Y is any $(n-1)$-connected space with $\pi_n Y \cong \mathbb{Z}$ and $Y\langle n\rangle \simeq (S^n\langle n\rangle)^\wedge_2$, and ι denotes a generator of $H^n(Y;\mathbb{Z})$, then $Y \simeq Y_{2,k}$ where k is the smallest integer such that $(2^k\iota)^2 = 0$.

At an odd prime p, the mod p reduction of ι_1 is an *exterior* generator detected in the mod p cohomology of $K(\mathbb{Z}/2, n-1)$, but it has non-trivial integral Steenrod operations, such as $\mathcal{P}^1\beta$, acting on it. This operation is represented by classes of order p in $H^{n+2p-1}(K(\mathbb{Z}, n);\mathbb{Z})$ corresponding to the pair $(\mathcal{P}^1\iota_n, \beta\mathcal{P}^1\iota_n)$ in mod p cohomology. This shows that if Y is any space with $\pi_n Y \cong \mathbb{Z}$ and $Y\langle n\rangle \simeq (S^n\langle n\rangle)^\wedge_p$, and ι denotes a generator of $H^n(Y;\mathbb{Z})$, then $Y \simeq Y_{p,k}$ where k is the smallest integer such that $\mathcal{P}^1(p^k\iota) = 0$. ∎

Theorem 6.6. *The extended Postnikov genus set $\overline{G}_{[n]}(S^n)$ of homotopy types of spaces Y such that $Y[n] \simeq K(\mathbb{Z}, n)$ and $Y\langle n\rangle \simeq S^n\langle n\rangle$ is uncountable, in bijection with $\prod_p \mathbb{N}_+$, where the product is taken over all primes.*

Proof. We apply Theorem 3.3 and the identification $B\operatorname{aut}(S^n\langle n\rangle) \simeq \prod B\operatorname{aut}(S^n\langle n\rangle^\wedge_p)$ obtained above. Theorem 6.3 shows that the set of unpointed homotopy classes $[K(\mathbb{Z}, n), B\operatorname{aut}(S^n\langle n\rangle)]$ is uncountable, in bijection with $\prod_p \mathbb{N}_+$ and it remains to identify the action of the finite group $\operatorname{Aut}(K(\mathbb{Z}, n)) \cong \mathbb{Z}/2$. But we have seen that it is trivial at each prime since -1 is a unit in the p-adic integers. Hence the action is trivial. ∎

Elaborating a little bit on Construction 6.4, one can explicitly construct all these fake spheres.

Construction 6.7. We identify $[K(\mathbb{Z}, n), B\operatorname{aut}(S^n\langle n\rangle)]$ with $\prod_p \mathbb{N}_+$. An element in this set is a sequence $K = (k_p)$ consisting either of a natural number or the base point $*$ for each prime p. For each such sequence consider the homotopy pull-back Y_K of the diagram

$$\prod_p Y_{p,k} \to \prod K(\mathbb{Z}, n) \xleftarrow{\Delta} K(\mathbb{Z}, n)$$

where the spaces $Y_{p,k}$ have been constructed in Example 6.4 and the second arrow is given by the diagonal inclusion. The homotopy fiber of the map $Y_K \to K(\mathbb{Z}, n)$ is the product $\prod_p S^n\langle n\rangle^\wedge_p \simeq$

$S^n\langle n\rangle$. The restriction to $B\,\mathrm{aut}(S^n\langle n\rangle^\wedge_p)$ yields $Y_{p,k}$ which is classified by k_p. This describes all spaces in $\overline{G}_{[n]}(S^n)$.

Thus we have a good handle on all these fake spheres S^n. What is so special about the good old S^n among them? The answer is in Theorem 4.5.

Proposition 6.8. *Let Y be a space such that $Y[n] \simeq K(\mathbb{Z}, n)$ and $Y\langle n\rangle \simeq S^n\langle n\rangle$. Then, if Y is a finite complex, Y has the homotopy type of S^n.* ∎

Finally we address the question of what happens when one changes the n-th Postnikov section for a higher one. The result will basically remain the same. An explicit computation would prove to be more difficult, but the concrete example of fake spheres we have produced serve equally well now.

Proposition 6.9. *Let Y be an element in the extended Postnikov genus $\overline{G}_{[n]}(S^n_p)$, and $m \geq n$. For any large enough prime p we have that $Y[m] \simeq S^n_p[m]$ and $Y\langle m\rangle \simeq S^n_p\langle m\rangle$.*

Proof. Since Y has been constructed so that $Y\langle n\rangle \simeq S^n\langle n\rangle^\wedge_p$, the same is true for a higher connected cover. The claim about the m-th Postnikov section follows by choosing $p > \dfrac{m-n+3}{2}$ so $\pi_* S^n$ has no p-torsion in degrees $< m$. ∎

This implies again that, for any m, there are uncountably many homotopy types of spaces which look like odd spheres through the eyes of the m-th Postnikov section and m-connected cover. We end the section with a related computation of the extended Postnikov genus of complex projective spaces.

Theorem 6.10. *The extended Postnikov genus set $\overline{G}_{[2n+1]}(\mathbb{C}P^n)$ is uncountable for any $n \geq 1$.*

Proof. Let $C = \mathbb{C}P^n[2n+1]$ be the $(2n+1)$-st Postnikov section of $\mathbb{C}P^n$. There are fibrations $K(\mathbb{Z}, 2n+1) \to C \to K(\mathbb{Z}, 2)$ and $S^{2n+1}\langle 2n+1\rangle \to \mathbb{C}P^n \to C$ where $S^{2n+1}\langle 2n+1\rangle$ decomposes as $\prod_p S^{2n+1}\langle 2n+1\rangle^\wedge_p$. Obstruction theory shows that $[\mathbb{C}P^n, \mathbb{C}P^n] \to [C, C]$ is bijective so that self-maps of $\mathbb{C}P^n$ and C are classified up to homotopy by their degrees in $H^2(-;\mathbb{Z})$. In particular, $\mathrm{Aut}(C) \cong \mathrm{Aut}(\mathbb{C}P^n) \cong \mathbb{Z}^\times = \{\pm 1\}$ has two elements.

According to Theorem 3.3, $\overline{G}_{[2n+1]}(\mathbb{C}P^n) = [C, B\,\mathrm{aut}(S^{2n+1}\langle 2n+1\rangle)]/\mathbb{Z}^\times$.

Let $c\colon C \to B\,\mathrm{aut}(S^{2n+1}\langle 2n+1\rangle)$ be the classifying map for the standard $\mathbb{C}P^n$. We have seen that $B\,\mathrm{aut}(S^{2n+1}\langle 2n+1\rangle)$ splits as a product $\prod_p B\,\mathrm{aut}(S^{2n+1}\langle 2n+1\rangle^\wedge_p)$, and so the classifying map c decomposes as a product $c = \prod_p c_p \circ \Delta$, where $\Delta\colon C \to \prod_p C$ is the diagonal map and $c_p\colon C \to B\,\mathrm{aut}(S^{2n+1}\langle 2n+1\rangle^\wedge_p)$. For any sequence $m = (m_p)_p \in \prod_p \mathbb{Z}$ of integers, let P^n_m be the space classified by $\prod c_p \circ \prod m_p \circ \Delta\colon C \to \prod B\,\mathrm{aut}(X_p)$. For example $\mathbb{C}P^n$ and $C \times S^{2n+1}\langle 2n+1\rangle$ correspond respectively to the constant sequences (1) and (0).

Consider now the restriction of one of the components $c_p \circ m_p$ to the fiber $K(\mathbb{Z}, 2n+1)$. The degree m_p map on $\mathbb{C}P^n$ induces the degree m_p^n map on the cover S^{2n+1}, so that this restriction corresponds to the class of m_p^n in the coset $\mathbb{Z}_p^\wedge/(\mathbb{Z}_p^\wedge)^\times$ we obtained in Theorem 6.3. In particular, for any choice $m_p = p^k$ this restriction represents a different homotopy class in $[K(\mathbb{Z}, 2n+1), B\operatorname{aut}(S^{2n+1}\langle 2n+1\rangle_p^\wedge)]$. In fact for any choice $k_p \in \mathbb{N}$, the sequences $m = (p^{k_p})$ yield an uncountable number of homotopy types of fake complex projective spaces. Indeed the spaces P_m^n are all distinct since the homotopy pull-back of $P_m^n \to C \leftarrow K(\mathbb{Z}, 2n+1)$ is homotopy equivalent to the fake sphere described by the sequence (nk_p) as in Construction 6.7. ∎

References

1. J. M. Alonso, *Fibrations that are cofibrations. II*, Proc. Amer. Math. Soc. **105** (1989), no. 2, 486–490.
2. V. Belfi and C. Wilkerson, *Some examples in the theory of P-completions*, Indiana Univ. Math. J. **25** (1976), no. 6, 565–576. MR 0418086 (54 #6130)
3. A. K. Bousfield, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra **9** (1976/77), no. 2, 207–220.
4. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 304.
5. W. G. Dwyer, *The centralizer decomposition of BG*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Progr. Math., vol. 136, Birkhäuser, Basel, 1996, pp. 167–184.
6. W. G. Dwyer, D. M. Kan, and J. H. Smith, *Towers of fibrations and homotopical wreath products*, J. Pure Appl. Algebra **56** (1989), no. 1, 9–28.
7. E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996.
8. L. Fuchs, *Infinite abelian groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970.
9. P. Hilton, *Homotopy theory and duality*, Gordon and Breach Science Publishers, New York, 1965.
10. ———, *On groups of pseudo-integers*, Acta Math. Sinica (N.S.) **4** (1988), no. 2, 189–192.
11. ———, *On the extended genus*, Acta Math. Sinica (N.S.) **4** (1988), no. 4, 372–382.
12. C. A. McGibbon, *A note on Miller's theorem about maps out of classifying spaces*, Proc. Amer. Math. Soc. **124** (1996), no. 10, 3241–3245. MR MR1328362 (96m:55033)
13. ———, *On the localization genus of a space*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Progr. Math., vol. 136, Birkhäuser, Basel, 1996, pp. 285–306. MR MR1397739 (97d:55019)
14. C. A. McGibbon and J. M. Møller, *Connected covers and Neisendorfer's localization theorem*, Fund. Math. **152** (1997), no. 3, 211–230. MR MR1444714 (98g:55017)
15. H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), no. 1, 39–87.
16. G. Mislin, *The genus of an H-space*, Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971), Springer, Berlin, 1971, pp. 75–83. Lecture Notes in Math., Vol. 249.
17. J. M. Møller and J. Scherer, *Can one classify finite Postnikov pieces?*, Bull. Lond. Math. Soc. **42** (2010), no. 4, 661–672. MR 2669687 (2012a:55020)
18. J.A. Neisendorfer, *Localization and connected covers of finite complexes*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 385–390.

19. J.J. Rotman, *An introduction to the theory of groups*, fourth ed., Graduate Texts in Mathematics, vol. 148, Springer-Verlag, New York, 1995.
20. J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7.
21. C. W. Wilkerson, *Applications of minimal simplicial groups*, Topology **15** (1976), no. 2, 111–130. MR MR0402737 (53 #6551)
22. Z. Wojtkowiak, *On fibrations which are also cofibrations*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 120, 505–512.
23. A. Zabrodsky, *On the genus of finite CW-H-spaces*, Comment. Math. Helv. **49** (1974), 48–64. MR MR0353308 (50 #5792)
24. ———, *On phantom maps and a theorem of H. Miller*, Israel J. Math. **58** (1987), no. 2, 129–143.
25. A. Zabrodsky, *p equivalences and homotopy type*, Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), Springer, Berlin, 1974, pp. 161–171. Lecture Notes in Math., Vol. 418. MR MR0377867 (51 #14036)

Matematisk Institut, Universitetsparken 5, DK–2100 København
E-mail address: moller@math.ku.dk
URL: htpp://www.math.ku.dk/~moller

École Polytechnique fédérale de Lausanne, CH-1015 Lausanne, Switzerland
E-mail address: jerome.scherer@epfl.ch
URL: http://hessbellwald-lab.epfl.ch/Scherer